International Journal of Scientific & Engineering Research Volume 2, Issue 10, Oct-2011
ISSN 2229-5518

On subgroups of a finite p-groups

A. D. Akinola

Abstract: In this paper we proved some theorems on normal subgroups, on-normal subgroup, minimal nonmetacyclic and maximal class of a p-group G.

1. INTRODUCTION

Let *G* be a finite p-group. Most authors have worked on p-groups among which we will mention few. Y. Berkovich [6] have worked on subgroups and epimorphic images of finite p-groups. Y. Berkovich [4] worked on finite p groups with few minimal nonabelian subgroups. Y.Berkovich [5] worked on subgroups of finite p-groups. Y. Berkovich [1] also woraked on abelian subgroup of a p-group *G*. Z. Janko [2,3] worked on element of order at most 4 in finite 2-groups and On finite nonabelian 2 groups all of whose minimal nonabelian subgroups are of exponent 4.In this paper we give an answer to some of the questions post by Y.Berkovich in [1].

2. DEFINITIONS

2.1 Definition

If a group *G* has order p^m where *p* is a prime number and *m* is a positive integer, then we say that *G* is a *p*-group.

2.2. Definition

Let *H* be a subgroup *G* and let *a 2 G*, the normalizer of *H* in *G* is denoted by *N*(*H*) defined by $\mathrm{N(H)} = \{\mathrm{a} \in \mathrm{G} : \mathrm{aHa}^{-1} = \mathrm{H}\}$. It follows that the normalizer of a subgroup *H* is the whole group *G* if and only if *H* is normal in *G.*

2.3. Definition

If $x \in G$, the centralizer of *x* in *G*, denoted by *CG*(*x*) is the set of all $a \in G$ that commute with *x* . i.e $CG(x) = \{a \in G : axa^{-1} = xg$ .It is immediate that *CG*(*x*) is a subgroup of *G*. Also $x \in CG(x)$.

2.4. Definition

A group *G* which contains a cyclic normal subgroup *A* such that *G/A* is also cyclic is a metacyclic group. Dihedral groups and generalized quaternion groups are examples of metacyclic groups.

2.5. Definition

A group *G* is said to be minimal nonmetacyclic if *G* is not metacyclic but all of its proper subgroups are metacyclic.

2.6. Definition

The length of lower central series of *G*, that is the greatest integer *c* for which $\gamma_c(G) > \{1\}$ is called the class of *G*. The class of a *p*-group is a measure of the extent to which the group is non abelian. Abelian group are of class 1 and
conversely group of class 1 are abelian.

2.7. 2.7 Definition

The group of order p^m and class *m*-1 for some $m \geq 3$, a *p*-group is said to be of maximal class where $(G : \gamma_2(G)) = p^2$; $\gamma_{i-1}(G) : \gamma_i(G)) = p$ $(i = 3; 4; ::::; m)$.

3. MAIN RESULT

3.1. Theorem

Suppose a p-group *G*,*p* > 2 contains an abelian self centralizer subgroup *A* of order p^3 and $|NG(A) : A| = p$. Then the number of such subgroup in *G* is congruent to 1(*modp*).

Proof:

For $H \leq G$, let $q_3(H)$ denote the number of self centralizer subgroup of order p^3

ISSN 2229-5518

contained in *H*. We have that $p^2 \equiv 1(\mathrm{mod}\, p)$ Let μ denote the set of all maximal subgroups of *G*.It is known that $|\mu| \equiv 1(\mathrm{mod}\, p)$. By hall's enumeration principle [7] $q_3(G) = \sum_{H\in\mu} q_3(H)(\mathrm{mod}\, p)$ Suppose that the theorem has proved fall proper subgroup of *G*. Take $H \in \mu$. By induction hypothesis $q_3(H) = 0$ or $q_3(G) \equiv 1(\mathrm{mod}\, p)$. If $q_3(G) = 1(\mathrm{mod}\, p)$ for all $H \in \mu$. Then by (1) $q_3(H) \equiv |\mu| \equiv 1(\mathrm{mod}\, p)$.Proving the theorem.

Therefore suppose we may assume that some maximal subgroup of *G*, say *H* has no abelian self centralizer subgroup of order p^3. Suppose that *H* contains a subgroup *L* of order p^4 and exponent *p*.Let *A* be a maximal abelian self centralizer subgroup of *L*.Since *A* < *L* and $C_L(A) = A$, it follows that |*A*| = p^3,contrary to the what was proved in the previous paragraph.Therefore *H* has no subgroup of order p^4 .

Suppose $|N_G(A) : A| = p$ this implies that *A* is a normal subgroup of *G*. Also assume that $N_G(A) = G$ then *A* is maximal in *G*.

Let $q'_3(H)$ be the number of normal abelian self centralizer subgroup of order p^3. in *G*. Since $q_3(G) = q'_3(G)(\mathrm{mod}\, p)$. it suffices to prove that $q'_3(G) \equiv 1(\mathrm{mod}\, p)$.

Therefore we may assume that *G* contains a normal abelian self centralizer subgroup K_1 of order p^3.,$K_1 \neq K$. Set $D = KK_1$. By fittings lemma, the nilpotency class of *D* is at most two. Therefore by [1] *exp* (*D*) = *p*. Considering $D \cap H$ and taking into account that *H* has no subgroups of order p^4 and exponent *p*, we conclude that $|D| = p^4$. By lemma 3 [1] $q_e(D) \equiv 1(\mathrm{mod}\, p)$. Hence the number of abelian normal self centralizer subgroup of order p^3. in *D* is congruent to 1 modulo *p*.

Assume that *G* contains a normal abelian self centralizer subgroup K_2 of order p^3 such that K_2 is not a subgroup of *D* with $K \cap K_1$ not a subgroup of K_2. It follows that $|K \cap K|_1 = p^2 = |K \cap K_2|$. Since $K \cap K_1, K_1 \cap K_2$ are different maximal subgroups of K_2. We conclude that $K_2 = (K \cap K_1)(K_1 \cap K_2) < KK_1 = D$ contrary to the choice of K_2 .Therefore such K_2 does not exist.

Therefore the number of maximal normal abelian self-centralizer subgroup of order p^3 in *G* is congruent to 1 modulo *p* .

3.2. Theorem

Let *A* be a subgroup of a p-group *G* such that $C_A(G)$. is metacyclic . If |*A*| =*p*, then *G* has normal subgroup of order p^{p+1}.and exponent *p*.

Proof:

We may assume that *A* < *Z*(*G*). By [8] $C_G(A) = N_G(A)$. since |*A*|=*p*. Suppose that *D* is a normal subgroup of *G* of exponent *p*. We may assume that $|D| > p^{p+1}$. and $|AD| > p^2$ .Then $C_A(D) > \{1\}$ .It follows that $H = AC_D(A) < C_G(A)$ that *H* is metacyclic. We have $C_{AD}(H) = H$ .. Therefore by [1] *AD* is of maximal class. This is a contradiction since *D* is not of maximal class. Therefore $|D| = p^{p+1}$. Hence the result.

3.3 Theorem

Suppose that *p*-group *G*, *p* = 2 contains an abelian normal subgroup of order p^{p+1}. Then the number of nonabelian, non normal subgroup of order p^{p+1} Is congruent to 0(*modp*).

Proof:

Let $H \leq G$ .. Let $q_3(H)$ denote the number of nonabelian normal subgroup of order p^{p+1}. contained in *H*. We have to prove that $q_3(H) \equiv 0(\mathrm{mod}\, p))$. Let μ denote the set of all maximal subgroups of *G*. It is known that $|\mu| \equiv 1(\mathrm{mod}\, p)$.

ISSN 2229-5518

Take $H \in \mu$. By induction hypothesis $q_3(H) \equiv 0(\bmod p))$. By [6] *H* contains one abelian normal subgroup of order p^{p+1}. Therefore $q_3(G) \equiv 0(\bmod p))$ proving the theorem.

Let $q'_3(H)$ be the number of nonabelian, non normal subgroup of order p^{p+1} in *G*. We may assume that *G* contains one abelian normal subgroup of order p^{p+1}. By [1] the number of subgroup of order p^{p+1} is congruent to 1(*modp*). Therefore $q'_3(G) \equiv 0(\bmod p)$. since by [1] *G* contains one abelian normal subgroup of order p^{p+1}.

3.4. Theorem

Let *G* be a *p*-group and suppose *N* is non normal subgroup of a *p*-group *G*. If *A* is a maximal non normal subgroup of *N* then $C_N(A) = Z(G)$.

Proof:

Assume that $C = C_N(A) > Z(G)$.Then $C = N \cap C_G(A)$.Let *B* be non normal subgroup of *N* such that *B/A* is a N/*A* non normal subgroup of exponent *p* in *C/A*.Then *B* is not normal in *G* and *B* > *A* contrary to the choice of *A* that *A* is maximal non normal subgroup of *N*. Therefore $C_N(A) = Z(G)$. Hence the result.

3.5. Theorem

Suppose that *p*-group *G* contained a subgroup *M* of maximal class such that $C_G(M) < M$ and $|M| > p^3$ where *p* = 2, then *G* is of maximal class.

Proof:

$|M| > p^3, C_G(M) = Z(M) = p$ since *M* is of maximal class.

Also $C_G(M) = Z(M) = Z(G) = p$

Therefore by [4] *G* is of maximal class since *Z*(*G*) = *p* which complete the proof .

3.6. Theorem

Let *A* < *N* < *G*, where *N* is a non normal subgroup of *G* and *A* is a maximal subgroup of *N*, $\exp(N) < p^n, p^n > 2$. Let μ be the set of all maximal non normal subgroup of *N* such that *exp*(*A*) < p^n.Then $|\mu| \equiv 0(\bmod p)$.

Proof:

Assume that *N* is a non normal subgroup of *G*. Also let *A* be a maximal subgroup of *N* .Let μ be the set of all maximal non normal subgroup of *N*. We have to prove that $|\mu| \equiv 0(\bmod p)$. By sylow's theorem, the number of subgroup of a group is congruent to 1(*modp*).

By [6] *N* contains one maximal normal subgroup which implies that the number of maximal non normal subgroup of *N* is congruent to 0(*modp*) . i.e $|\mu| \equiv 0(\bmod p)$.

3.7. Theorem

Let $A < B \le G$. where *B* is a nonabelian subgroup of a non abelian *p*-group *G*, *exp*(*B*) $\le p^m$. and $p^m > 2$, *p* = 2;*m* > 2. Let μ be the set of all non abelian subgroup *T* of *G* such that *A* < *T*, $|T : A| = p^2$ and $\exp(T) = p^m$. Then $|\mu| \equiv 0(\bmod p)$.

Proof:

Let *G* be a 2- group of order 2^m.Let *G* be member of subgroups of *G* of order $2^n n < m$.such that *T* is non abelian.

Let μ be the set of all nonabelian subgroup *T* of *G*.

Let *A* be member of subgroup of *G* such that $|T : A| = p^2$. By sylow's theorem, the number of subgroup of a group *G* is congruent to 1(*modp*).

If $|T : A| = p^2$ then $|A| = p^{n-2}$

By [6], for every value of *n; n* < *m, G* contains one abelian subgroup T'' of order p^n with $|T'' : A| = p^2$. Therefore the number of *T* such that *A* < *T* and $|T : A| = p^2$ is congruent to 0(*modp*). Hence the result.

References

[1] Y. Berkovich, On Abelian subgroups of *p*-groups,J. of Algebra 199,262-

280 (1998).
[2] Z.Janko,Elements of order at most 4 in finite 2-group, J. Group theory 8 (2005),683-686
[3] Z. Janko,On finite nonabelian 2-groups all of whose minimal nonabelian subgroups are of exponent 4, J. Algebra 315 (2007) 801-808
[4] Y. Berkovich, Finite *p*-groups with few minimal nonabelian subgroups, J.Algebra 297 (2006) 62-100.
[5] Y. Berkovich, On subgroups of finite *p*-groups, J. Algebra 224,(2000),198-240.
[6] Y. Berkovich, On subgroups and Epimorphic images of finite *p*-Groups ,J. Algebra 248 (2002),472-553.
[7] P.Hall, A contribution to the theory of groups of prime power order, Proc. London Math. Soc. (2) 36, (1933),29-95.
[8] Y.Berkovich, Groups with a cyclic subgroups of index *p*,frattini subgroups, pre-print.

A. D. Akinola,
Mathematics Department,
College of Natural Sciences,
University of Agriculture,
Abeokuta,Ogun State,
Nigeria.